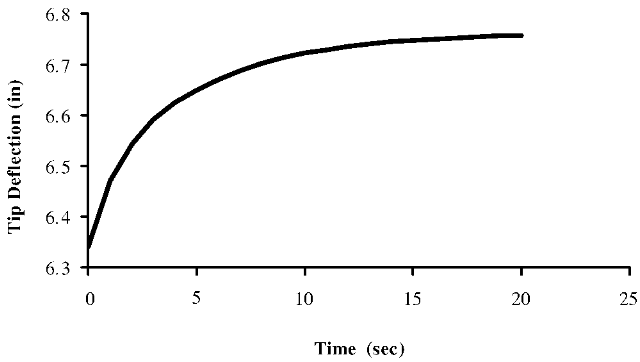

Fig. 4 Deflection history for laminated cantilevered beam.

which, in terms of Lamé constants, is

$$E_f = \frac{1}{8}\left[\frac{\mu_1(3\lambda_1 + 2\mu_1)}{\lambda_1 + \mu_1} + \frac{7\mu_2(3\lambda_2 + 2\mu_2)}{\lambda_2 + \mu_2}\right]$$

When the preceding procedure is followed and the CP used, the deflection at $x = 0$ is given as

$$v(0, t) = \frac{PL^3}{3I}\left[\frac{1}{E_f} + \frac{8\gamma_1}{d}\sum_{0}^{4}\frac{g(\alpha_k)}{p'(\alpha_k)}e^{\alpha_k t}\right]$$

where d is a constant and $g(s)$ and $p(s)$ are polynomials. The constants α_k are the roots of $p(s) = 0$ with $\alpha_1 = 0$. For a sample calculation (using properties obtained from Ref. 5, with the subscript 1 representing the composite properties and 2 being the balsa wood properties), the solution is shown in Fig. 4: $E_1 = 4E6$ psi ($27.6E9$ Pa), $E_2 = 10E3$ psi ($68.9E6$ Pa), $\nu_1 = 0.14$, $\nu_2 = 0.011$, $\tau_{11} = 5$ s, $\tau_{12} = 2.5$ s, $\tau_{21} = 1.0$ s, $\tau_{22} = 0.5$ s, and $\gamma_1 = 1.667E{-8}$ psi ($0.11E{-3}$ Pa).

Conclusions

A study has been undertaken to determine the viscoelastic parameters by using the CP. The CP was applied to the elastic solution for the deflection of a cantilevered beam to obtain the resulting viscoelastic equation. The Kelvin–Voigt model was then used to express Lamé constants in terms of the viscoelastic parameters. The equation was Laplace transformed to express the deflection equation in terms of length and time. The resulting equation was then plotted and adjusted to fit curves obtained from previous studies.[5,7] From the curve fitting, the viscoelastic parameters were extracted.

Further work in this area will produce experimental results that will be used to determine the viscoelastic constants, and these will then be confirmed by the use of the CP. The main attraction of this method is that the extracted viscoelastic constants are determined only once and then can be used in the analysis of more complicated structures as long as the same material system is used.

Acknowledgments

The funds for this study were generously provided as a grant from the Office of Naval Research (Contract N00001499C0440) by Project Manager Patrick Potter, in support of Mississippi State University's lead in the Southeastern University Consortium's participation in the Composite Helicopter Hangar Program. Also, the authors are indebted to Z. U. A. Warsi, whose expert knowledge and guidance have been necessary in the formulation of the problem.

References

[1]Eringen, A. C., *Mechanics of Continua*, Wiley, New York, 1967, pp. 368–370.

[2]Lee, E. H., "Stress Analysis in Viscoelastic Bodies," *Quarterly of Applied Mathematics*, Vol. 13, 1955, pp. 183–190.

[3]Biot, M. A., "Linear Thermodynamics and the Mechanics of Solids," *Proceedings of the Third U.S. National Congress of Applied Mechanics*, American Society of Mechanical Engineers, New York, 1958, pp. 1–18.

[4]Schapery, R. A., "Stress Analysis of Viscoelastic Composite Materials," *Journal of Composite Materials*, Vol. 1, 1967, pp. 228–267.

[5]Palmer, D., Adamchak, J., and Sorathia, U., "Structural Integrity of Composites at Elevated Temperatures," U.S. Naval Surface Warfare Center, Carderock Div., Rept. NSWCCD-65-TR-1999/17, 1999.

[6]Ha, S. K., and Springer, G. S., "Time Dependent Behavior of Laminated Composites at Elevated Temperatures," *Journal of Composite Materials*, Vol. 23, 1989, pp. 1159–1197.

[7]Critchfield, M. O., and Palmer, D., "Viscoelastic Behavior of Composites," U.S. Naval Surface Warfare Center, Carderock Div., 2001.

[8]Zocher, M. A., Groves, S. E., and Allen, D. H., "A 3-D FE Formulation for Thermoviscoelastic Orthotropic Media," *International Journal for Numerical Methods in Engineering*, Vol. 40, 1997, pp. 2267–2288.

[9]Gibson, R. F., *Principles of Composite Material Mechanics*, McGraw–Hill, New York, 1994, pp. 192–194.

A. N. Palazotto
Associate Editor

Curvature Corrections for Algebraic Reynolds Stress Modeling: A Discussion

Antti Hellsten*
Helsinki University of Technology,
FIN-02015 HUT, Finland

I. Introduction

TURBULENT flows are known to be sensitive to streamline curvature. It is also clear that linear eddy-viscosity turbulence models (EVMs) are completely insensitive to the effects of streamline curvature. On the other hand, differential Reynolds stress models (RSMs) are able to capture these effects. However, RSMs have not become a standard tool in practical fluid-flow simulations, owing to their complexity, large computational work load, and typically unfavorable effect on numerical stability. Therefore, explicit algebraic Reynolds stress models (EARSMs) have become increasingly popular during recent years. EARSMs are two-equation models sharing much of the computational manageability of the EVMs while partially retaining the more realistic physical background of the underlying RSM. However, the sensitivity to the streamline curvature is partially lost through the weak-equilibrium assumption invoked to derive algebraic RSMs. It has been shown by several authors that, in principle, this deficiency can be alleviated by assuming the weak equilibrium in a suitable curvilinear stream-following coordinate system.

The algebraic RSM (ARSM) formulation for curved flows will be revisited. This is to show how the weak equilibrium assumption can be invoked in a suitable curvilinear coordinate system to minimize approximately the resulting error for curved flows. Such approximations are proposed in the literature based on the rotation rate of 1) the velocity vector,[1] 2) the acceleration vector,[2,3] and 3) the principal system of the strain-rate tensor.[3–6] The first approach is, in principle, not fully generalizable because of its lack of Galilean invariance.

The purpose of this Note is to show that the acceleration method is also generally invalid. The observed singular behavior of the acceleration method is discussed in general, and its failure is numerically demonstrated in a plane duct flow including a 180-deg bend with a small radius of curvature.[7,8] The behavior of the acceleration method will be compared with that of the strain-rate and velocity-based methods.

Received 21 December 2001; revision received 29 April 2002; accepted for publication 13 May 2002.

*Research Scientist, Laboratory of Aerodynamics, P.O. Box 4400; Antti.Hellsten@hut.fi. Member AIAA.

II. ARSM Formulation for Curved Flows

A general quasilinear RSM written for the anisotropy tensor $\boldsymbol{a} = a_{ij} = \overline{u_i u_j}/k - \frac{2}{3}\delta_{ij}$ reads

$$\tau\left(\frac{\mathrm{D}\boldsymbol{a}}{\mathrm{D}t} - \mathcal{D}^{(a)}\right)\boldsymbol{a} = A_0\left[\left(A_3 + A_4\frac{P}{\varepsilon}\right)\boldsymbol{a} + A_1\boldsymbol{S} - (\boldsymbol{a}\boldsymbol{\Omega} - \boldsymbol{\Omega}\boldsymbol{a}) + A_2\left(\boldsymbol{a}\boldsymbol{S} + \boldsymbol{S}\boldsymbol{a} - \frac{2}{3}\mathrm{tr}\{\boldsymbol{a}\boldsymbol{S}\}\boldsymbol{I}\right)\right] \tag{1}$$

$\boldsymbol{S}$ and $\boldsymbol{\Omega}$ are the strain-rate and vorticity tensors, that is, the symmetric and skew-symmetric parts of the velocity gradient. Turbulent time scale is $\tau = k/\varepsilon$, and $A_0, \ldots, A_4$ are the RSM coefficients.

The corresponding ARSM is obtained in the weak equilibrium limit where the left-hand side of Eq. (1) is ignored, that is, the advection and diffusion of $\boldsymbol{a}$ are dropped. Note that the effect of ignoring the advection term $\mathrm{D}\boldsymbol{a}/\mathrm{D}t$ depends on the choice of coordinate system for representing the components of $\boldsymbol{a}$.

As suggested by, for example, Rodi and Scheuerer,[1] and later pointed out by Girimaji,[2] Gatski and Jongen,[5] and Wallin and Johansson,[3] the best ARSM for curved flows is obtained by ignoring the anisotropy advection in a suitable curvilinear coordinate system that follows the flow in some sense. To understand this, let us first consider the advection of $\boldsymbol{a}$ in any curvilinear coordinate system. Consider the anisotropy tensor transformed into the curvilinear system $\boldsymbol{T}\boldsymbol{a}\boldsymbol{T}^t$, where $\boldsymbol{T}$ is the transformation matrix from the inertial Cartesian background frame to the curvilinear system. Note that the transformation matrix varies in space. Next, we take the material derivative $\mathrm{D}/\mathrm{D}t$ of $\boldsymbol{T}\boldsymbol{a}\boldsymbol{T}^t$ and transform it back to the Cartesian background frame as shown by Wallin and Johansson[3] in a manner similar to that of Gatski and Jongen[5]:

$$\boldsymbol{T}^t\frac{\mathrm{D}}{\mathrm{D}t}(\boldsymbol{T}\boldsymbol{a}\boldsymbol{T}^t)\boldsymbol{T} = \frac{\mathrm{D}\boldsymbol{a}}{\mathrm{D}t} + \boldsymbol{a}\frac{\mathrm{D}\boldsymbol{T}^t}{\mathrm{D}t}\boldsymbol{T} + \boldsymbol{T}^t\frac{\mathrm{D}\boldsymbol{T}}{\mathrm{D}t}\boldsymbol{a} \tag{2}$$

When the following notation is used,

$$\frac{\mathrm{D}\boldsymbol{T}^t}{\mathrm{D}t}\boldsymbol{T} = -\boldsymbol{T}^t\frac{\mathrm{D}\boldsymbol{T}}{\mathrm{D}t} = \boldsymbol{\Omega}^{(r)} \tag{3}$$

the material derivative of $\boldsymbol{a}$ can be expressed as

$$\frac{\mathrm{D}\boldsymbol{a}}{\mathrm{D}t} = \boldsymbol{T}^t\frac{\mathrm{D}}{\mathrm{D}t}(\boldsymbol{T}\boldsymbol{a}\boldsymbol{T}^t)\boldsymbol{T} - \left(\boldsymbol{a}\boldsymbol{\Omega}^{(r)} - \boldsymbol{\Omega}^{(r)}\boldsymbol{a}\right) \tag{4}$$

In principle, the optimal ARSM can be obtained by dropping only the differential part of the material derivative, $\boldsymbol{T}^t[\mathrm{D}(\boldsymbol{T}\boldsymbol{a}\boldsymbol{T}^t)/\mathrm{D}t]\boldsymbol{T}$. The latter term in Eq. (4) can be included in the ARSM, in fact, by only modifying the vorticity tensor because $\boldsymbol{\Omega}^{(r)}$ is of similar tensorial form as $\boldsymbol{\Omega}$ (skew symmetry). Redefining the vorticity tensor as

$$\boldsymbol{\Omega}^* = \boldsymbol{\Omega} - (\tau/A_0)\boldsymbol{\Omega}^{(r)} \tag{5}$$

corresponds to assuming the weak equilibrium in the given curvilinear system. The vorticity modification owing to curvature $\boldsymbol{\Omega}^{(r)}$ is related to the rotation rate vector $\boldsymbol{\omega}^{(r)}$ of the local basis of the curvilinear system by $\Omega^{(r)}_{ij} = -\epsilon_{ijk}\omega^{(r)}_k$.

III. Proposed Methods

The problem of formulating a suitable ARSM for flows with streamline curvature is now reduced to finding a suitable curvilinear system in which the weak-equilibrium assumption is best satisfied. In principle, the optimal system is such where the ignored term, $\boldsymbol{T}^t[\mathrm{D}(\boldsymbol{T}\boldsymbol{a}\boldsymbol{T}^t)/\mathrm{D}t]\boldsymbol{T}$, in Eq. (4) is minimized. Such an optimal coordinate system that really minimizes the effect of the weak-equilibrium assumption cannot be found in closed form within the framework of EARS modeling. Instead, a suitable approximation for it must be found. The methods proposed in the literature are discussed hereafter.

A. Streamline Coordinate System

Rodi and Scheuerer[1] used the streamline system in their curved boundary-layer computations. In computations of more general flowfields, this means equating $\boldsymbol{\omega}^{(r)}$ with the rotation rate of a local basis attached to the velocity vector in the selected coordinate system. Galilean invariance, which is commonly required from rigorously derived turbulence models, is not satisfied. Therefore, other alternatives have been proposed by various authors to replace the streamline system.

B. Acceleration Coordinate Systems

Girimaji[2] proposed using a local frame that follows the acceleration vector $\dot{\boldsymbol{U}} = \mathrm{D}\boldsymbol{U}/\mathrm{D}t$. This is motivated by acceleration being a Galilean invariant quantity. Girimaji only discussed two-dimensional applications. More recently, Wallin and Johansson[3] proposed an approximate method for three-dimensional flows

$$\boldsymbol{\omega}^{(r)} \approx \boldsymbol{\omega}^{(\mathrm{W-J})} = \frac{\dot{\boldsymbol{U}} \times \ddot{\boldsymbol{U}}}{\dot{\boldsymbol{U}} \cdot \dot{\boldsymbol{U}}} \tag{6}$$

Although not pointed out by Wallin and Johansson,[3] Eq. (6) can easily be shown to be equal to the exact form in two-dimensional mean flows. Therefore, the behavior of the acceleration basis is studied using Eq. (6) in this Note.

In circular flows, the acceleration vector is orthogonal to the velocity vector. Therefore, the rate of change of the acceleration basis equals the rate of change of the streamline basis in a fixed background frame. Thus, ARSM derived in the acceleration frame is equal to the one derived in a streamline frame, provided that the velocity of the background coordinate system is chosen so that the flow is circular. Girimaji[2] stated that every flow can be considered as locally circular because a suitable Galilean transformation can be made for the local frame to arrive at locally circular flow, that is, to make the velocity and acceleration vectors orthogonal. The purpose of this Note is to point out the central failure behind this reasoning. That is, although the flow may formally be seen as locally circular from a certain acceleration basis, it does not imply that such acceleration system is suitable for the weak-equilibrium assumption. As a matter of fact, the acceleration basis may behave in a very wild manner and even become singular in certain circumstances. The velocity and the radius of curvature of locally circular flow correspond to the rotation rate of the local acceleration basis. Singularity formally corresponds to the limit of the infinite rotation rate of the acceleration system and, thus, the vanishing radius of curvature of the locally circular flow in that system.

As an example, let us consider a slightly curved flow that is first accelerating and then, at a certain point, begins to decelerate (or vice versa). If the radius of curvature of the streamlines is large enough, the velocity and the acceleration vectors are almost aligned with each other. This is a typical situation in flows over slightly curved surfaces, for example, on the upper surface of a wing. If the radius of curvature is roughly constant, the basis of the curvilinear system should also change at about constant rate. The acceleration basis, however, singularly turns about 180 deg at the location where the streamwise acceleration changes its sign. In contrast, there is no reason for the anisotropy components to change rapidly in such a situation.

Starting and ending curvature are further examples of singularities of the acceleration base. At the very instant when a fluid particle enters or exits any curved part of a streamline, the acceleration vector immediately rotates to a new orientation, which may be almost orthogonal to its former orientation. Also, the anisotropy tensor may possibly change its orientation quite rapidly in this kind of situation, but hardly as fast as the acceleration vector.

C. Principal Axes of the Strain-Rate Tensor

Spalart and Shur proposed using the rate of change of the strain rate tensor $\dot{\boldsymbol{S}}$ as a measure of the rotation and curvature effects with the aim of correcting EVMs.[4] The rationale, in two-dimensional flows, was to identify the rate of change of the direction of the principal axes of the strain rate tensor $\mathrm{D}\alpha/\mathrm{D}t$. This measure gives the rotation rate $\omega_i^{(r)}$ in the direction normal to the plane of the flow as

$$\omega_3^{(r)} \approx \omega_3^{(\mathrm{S-S})} = \frac{S_{11}\dot{S}_{12} - S_{12}\dot{S}_{11}}{2\left(S_{11}^2 + S_{12}^2\right)} \tag{7}$$

This measure for two-dimensional mean flows was also derived by Gatski and Jongen.[5] It is expected to model the rotation rate of the optimal system quite well because the material derivatives of the strain-rate and anisotropy tensors are typically rather closely related with each other in the weak-equilibrium limit where $\boldsymbol{a} = f(\boldsymbol{S}, \boldsymbol{\Omega})$ and, furthermore, the leading term of anisotropy is directly proportional to $\boldsymbol{S}$. Based on this reasoning, Eq. (7) is expected to be the best available approximation for $\omega_3^{(r)}$. It is Galilean invariant, and, very recently, Wallin and Johansson derived a new, general strain-rate based method that reduces to Eq. (7) in two-dimensional mean flows.[3] They also showed that the original three-dimensional extension of Eq. (7) proposed by Spalart and Shur[4] is inconsistent with helical flows where the axial velocity component varies with the radial coordinate.

IV. Numerical Example

Flow in a 180-deg U-duct[7,8] is an illustrative example containing local flow situations where the acceleration basis behaves unfavorably and even goes singular. First, an a priori test of Eqs. (6), (7), and the streamline method is performed. A priori means here that the rotation rate $\omega_3^{(r)}$ is evaluated from a frozen flowfield computed using the noncorrected EARSM of Wallin and Johansson.[9]

The $\omega_3^{(r)}$ distributions are shown in Fig. 1. The plots reveal significant differences between each method. The angular velocity of the acceleration vector locally obtains of the order of 100 times higher values than what Eq. (7) gives near the end of the curved duct. Furthermore, an unexpected area of relatively high negative (clockwise) rotation is observed downstream of the bend in an area with almost zero curvature. This is where the acceleration vector suddenly turns around. Surprisingly, the acceleration system behaves quite smoothly in the beginning of the curvature.

Next, the flowfield is really computed using the curvature corrections on the same EARSM. As expected, the acceleration method (6) overestimates the curvature effects so strongly that no steady-state solution can be obtained. Instead, the flow tends to oscillate owing to the strong laminarizing influence of high $\omega_3^{(r)}$. The strain-rate method (7) and the streamline method allowed the solution to converge, although the convergence rate slows down remarkably owing to the reduced damping effect of the turbulence model. The predicted $\omega_3^{(r)}$ distributions are also presented in Fig. 1. The exaggerated effect of the acceleration method amplifies itself, mostly near the end of the curved duct, resulting in much higher values of $\omega_3^{(r)}$ than those observed in the a priori test. Unlike Eq. (6), the strain-rate method (7) and the streamline method predicted quite similar $\omega_3^{(r)}$ distributions to those seen in the a priori tests.

A detailed comparison of the computed results with experimental data is beyond the scope of this Note. The effect of the curvature corrections in the results is being assessed at the time of preparing this Note.[10]

Given these observations, no generalizability can be expected from any curvature corrections based on the coordinate system attached to the acceleration vector.

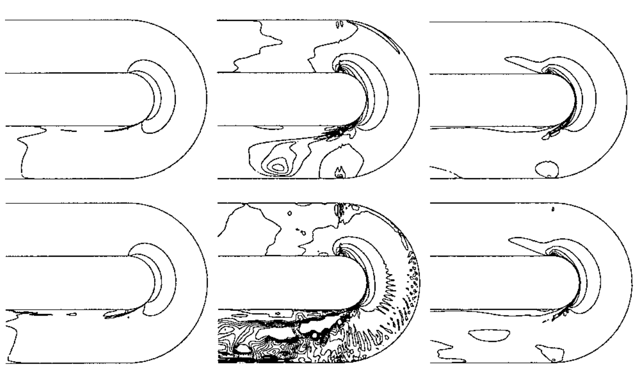

Fig. 1 A priori test (upper row) and actually predicted rotation rate $\omega_3^{(r)}$ (lower row) according to the streamline method (left), the acceleration method (6) (middle), and the strain-rate method (7) (right); the flow direction is from up to down.

V. Conclusions

The ARSM formulation for curved flows is revisited. This is to show how the weak-equilibrium assumption can be invoked in a suitable curvilinear coordinate system to minimize approximately the resulting error for curved flows. Such approximations are proposed in the literature based on 1) the streamline system, 2) the acceleration system, and 3) the strain-rate system. The first one of these has not been commonly accepted because of its lack of Galilean invariance.

The central idea of this Note is to show that the acceleration method is also generally invalid. The observed singular behavior of the acceleration basis is discussed in general, and its failure is numerically demonstrated in a plane U-duct flow. The behavior of the acceleration method is compared with that of the strain-rate and streamline based methods. The strain-rate based method shows promising behavior.

Acknowledgments

This work has partially been carried out within the High Level Modeling of High Lift Aerodynamics (HiAer) project. The HiAer project is a collaboration between DLR, ONERA, the Royal Institute of Technology (Sweden), Helsinki University of Technology, Technical University of Berlin, Alenia, European Aeronautic Defence Co. EADS Airbus, QinetiQ, and the Swedish Defence Research Agency (FOI). The project is managed by FOI and is partially funded by the European Union (Project Ref: G4RD-CT-2001-00448). This work is also supported by the Finnish Technology Agency (TEKES). The support from TEKES for this research is gratefully acknowledged. The author also wishes to thank Stefan Wallin (FOI) for fruitful discussions.

References

1 Rodi, W., and Scheuerer, G., "Calculations of Curved Shear Layers with Two Equation Turbulence Models," *Physics of Fluids*, Vol. 26, 1983, pp. 1422–1436.

2 Girimaji, S., "A Galilean Invariant Explicit Algebraic Reynolds Stress Model for Turbulent Curved Flows," *Physics of Fluids*, Vol. 9, 1997, pp. 1067–1077.

3 Wallin, S., and Johansson, A., "Modelling Streamline Curvature Effects in Explicit Algebraic Reynolds Stress Turbulence Models," *International Journal of Heat and Fluid Flow*, Vol. 23, No. 5, 2002, pp. 721–730.

4 Spalart, P., and Shur, M., "On the Sensitization of Turbulence Models to Rotation and Curvature," *Aerospace Science and Technology*, No. 5, 1997, pp. 297–302.

5 Gatski, T., and Jongen, T., "Nonlinear Eddy Viscosity and Algebraic Stress Models for Solving Complex Turbulent Flows," *Progress in Aerospace Sciences*, Vol. 36, No. 8, 2000, pp. 655–682.

6 Rumsey, C. L., Gatski, T., and Morrison, J., "Turbulence Model Predictions of Strongly Curved Flow in a U-Duct," *AIAA Journal*, Vol. 38, No. 8, 2000, pp. 1394–1402.

7 Monson, D., Seegmiller, H., McConnaughey, P., and Chen, Y., "Comparison of Experiment with Calculations Using Curvature-Corrected Zero and Two-Equation Turbulence Models for a Two-Dimensional U-Duct," AIAA Paper 90-1484, June 1990.

8 Monson, D., and Seegmiller, H., "An Experimental Investigation of Subsonic Flow in a Two-Dimensional U-Duct," NASA TM 103931, 1992.

9 Wallin, S., and Johansson, A., "A Complete Explicit Algebraic Reynolds Stress Model for Incompressible and Compressible Turbulent Flows," *Journal of Fluid Mechanics*, Vol. 403, 2000, pp. 89–132.

10 Hellsten, A., Wallin, S., and Laine, S., "Scrutinizing Curvature Corrections for Algebraic Reynolds Stress Models," AIAA Paper 2002-2963, June 2002.

W. J. Devenport
Associate Editor